Modelling and Solving of Multidimensional Auctions[1]

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Abstract

Auctions are important market mechanisms for the allocation of goods and services. In the paper a complex trading model is proposed. The proposed model, called multidimensional auction, is a generalization of an LP model of standard single item English auction. Multidimensional auctions include the possibility of auction generalization for multi-item, multi-type, multi-round and multi--criteria. The multi-item model for multi-type auction is modelled and solved by multi-round approach with multiple criteria. The proposed model illustrates the possibility to formulate and solve multidimensional auctions as mathematical programming problems. Allowing bidders more fully to express preferences leads to improved economic efficiency and greater auction revenue.

Keywords: *auctions, combinatorial auctions, iterative methods, multiple criteria*

JEL Classification: D44, C61

Introduction

Auctions are important market mechanisms for the allocation of goods and services. Auctions are preferred often to other common processes because they are open, quite fair, and easy to understand by participants, and lead to economically efficient outcomes. Many modern markets are organized as auctions. Design of auctions is a multidisciplinary effort made of contributions from economics, operations research, informatics, and other disciplines. Auction theory has caught tremendous interest from both the economic side as well as the Internet industry. An auction is a competitive mechanism to allocate resources to buyers based on predefined rules. These rules define the bidding process, how the winner is determined, and the final agreement.

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Classification of auctions is based on some specific characteristics as:

- the numbers of sellers and buyers,
- traded items (indivisible, divisible, pure commodities, structured commodities),
- the number of items,
- the number of units of items,
- type of auctions (forward, reverse, double),
- evaluating criteria,
- preferences of the participants,
- complexity of bids (simply, related bids),
- organisation of auctions (single-round, multi-round, sequential, parallel, price schemes).

The literature concerning auctions is quite rich. The standard models are based on game theory (Klemperer, 2004; Krishna, 2002; Milgrom, 2004). The popularity of auctions and the requirements of e-business have led to growing interest in the development of complex trading models (see Bellosta et al., 2004; Bichler, 2000; Oliveira, Fonsesca and Steiger-Garao, 1999). Combinatorial auctions (see Cramton, Shoham and Steinberg, 2006; de Vries and Vohra, 2003) are those auctions in which bidders can place bids on combinations of items, so called bundles. The advantage of combinatorial auctions is that the bidder can more fully express his preferences (Sandholm and Boutilier, 2006). This is particular important when items are complements. The auction designer also derives value from combinatorial auctions. Allowing bidders more fully to express preferences often leads to improved economic efficiency and greater auction revenues. However, alongside their advantages, combinatorial auctions raise a host of questions and challenges.

The literature concerning applications of auctions is versatile. For example, auctions have been proposed for the distribution of airport arrival and departure time slots (Rassenti, Smith and Bulfin, 1982), have been used for allocating radio spectrums for wireless communications services (Cramton, 2002), truckload transportation (Caplice and Sheffi, 2006), bus routes (Cantillon and Pesendorfer, 2006), and industrial procurement (Bichler et al., 2006). Also in the Czech and Slovak literature, it is possible to find articles with the applications of auctions, such (Fiala and Šauer, 2011; Šauer et al., 1998) dealing with the use of auctions for the selection of coalition projects for pollution reduction and the paper (Klátik, Sičáková-Beblavá and Beblavý, 2013) examines use e-auctions as an allocation mechanism in public procurement in Slovak public sector.

The main contribution of the paper is to propose a general model of auctions, covering all aspects of multidimensional auctions together, and a solution procedure for this general model. This is based on several specific auction models that have only some of specific features of multidimensional auctions. These models were modified and merged into the general model. The complex multidimensional auction trading model is based on a linear programming (LP) model and its extensions. Iterative combinatorial multi-type auctions with multiple criteria are proposed in the paper. A solution procedure based on changes of criteria aspiration levels is proposed and presented.

The rest of the paper is organized as follows. Section 1 presents a LP model of single item auctions. Section 2 summarizes the extensions of the basic model as multidimensional auctions. In Section 3, a model of multi-item combinatorial auctions is formulated. A model of multi-type auctions is analysed in the Section 4. Multi-round auctions as a solution approach are presented in Section 5. Multi-criteria auctions are formulated in Section 6 and last Section presents conclusions.

1. Modelling of Single item Auctions

The English auction is an open auction with rising prices. The auctioneer starts the auction with low price, which gradually increases. The auction ends when no buyer is willing to increase the offer. Buyer wins the auction with the highest bidder will pay the highest offer. The English auction is commonly used for selling goods, most prominently antiques and artwork.

The problem is formulated as follows: Suppose that the seller offers one object to n potential buyers B_1, B_2, ..., B_n. Each buyer also has its own evaluation of the object v_i, which is the maximum that the buyer is willing to pay for the item. The buyer prefers to pay less. The aim is for a given set of bids find an item allocation that maximizes seller's revenue. Seller will not necessarily receive the maximal value v_i.

The problem can be formulated as follows:

v_i – the value of the item to the buyer i,
x_i – a binary variable indicating whether the item is assigned to buyer i (x_i = 1).

$$\sum_{i \in N} v_i x_i \rightarrow \max$$

subject to
$$\sum_{i \in N} x_i \leq 1 \tag{1}$$
$$x_i \in \{0, 1\}, \ \forall i, i \in N = \{1, 2, \dots, n\}$$

The objective function expresses the target, i. e. maximize the revenue of the seller. Constraint indicates that no more than one buyer receives the item.

Solving the problem (1) is one way of determining the winner, if the maximal values of the item for buyers are known.

However, the buyers may prefer English auction with gradual increasing of prices because it allows them to get the item without paying the maximal value.

Consider the LP relaxation of the problem (1):

$$\sum_{i\in N} v_i x_i \to \max$$

subject to
$$\sum_{i\in N} x_i + y = 1 \tag{2}$$
$$x_i \le 1,\ \forall i,\ i \in N = \{1, 2, \ldots, n\}$$
$$y \ge 0,\ x_i \ge 0,\ \forall i,\ i \in N = \{1, 2, \ldots, n\}$$

In this formulation, a new variable y is added that represents the seller not assigning the item to any buyer. The constraints are dropped because the problem (2) has integer optimal solution.

The corresponding dual problem to the problem (2)

$$p + \sum_{i\in N} u_i \to \min$$

subject to
$$p + u_i \ge v_i\ \forall i,\ i \in N = \{1, 2, \ldots, n\} \tag{3}$$
$$p \ge 0,\ u_i \ge 0,\ \forall i,\ i \in N = \{1, 2, \ldots, n\}$$

The dual variable p can be interpreted as the selling price of the item, the dual variables u_i represent utilities for buyers as difference of the evaluation and the price of the item

$$u_i = \max\,[0, v_i - p]$$

The primal solution ($\mathbf{x}$, y) and the dual solution (p, $\mathbf{u}$) are optimal for a pair of problems (2) and (3) if complementary slackness conditions hold:

$$p > 0 \Rightarrow \sum_{i\in N} x_i + y = 1 \tag{4}$$

$$u_i > 0 \Rightarrow x_i = 1\ ,\ \forall i,\ i \in N = \{1, 2, \ldots, n\} \tag{5}$$

$$x_i > 0 \Rightarrow u_i = v_i - p,\ \forall i,\ i \in N = \{1, 2, \ldots, n\} \tag{6}$$

$$y > 0 \Rightarrow p = 0 \tag{7}$$

The conditions are then interpreted as follows:

- Someone gets the item if the price p is positive. It follows from the conditions (4) and (7).
- Any buyer not receiving the item must have utility $u_i = 0$, i.e. the price p is greater than his valuation v_i. It follows from the condition (5).
- By the condition (6), if the buyer i gets the item, then the selling price $p = v_i - u_i$.

Primal-dual simplex algorithm tries to achieve primal and dual admissibility by gradual elimination of violations of the complementary slackness conditions. This corresponds exactly to the English auction process. In this sense, the auction is nothing other than a LP solver. The dual variable p is interpreted as the current price of the item and the dual variables u_i express the corresponding utilities for buyers. The primal variable x_i indicates provisional winner of the auction arbitrarily selected among buyers who are interested in the item at the price p. The primal variable y indicates if no buyer is interested in the object at the given price p. When the complementary slackness conditions hold, so the auction ends and an optimal solution is found. During the auction in each round, the conditions (4), (6) and (7) hold. The condition (5) might not hold since there may be non-winners whose valuations are greater than the continuous price. Primal-dual algorithm works by increasing p until the condition (5) holds, i.e. to achieve a single winner of the auction.

2. Multidimensional Auctions

An auction provides a mechanism for negotiation between buyers and sellers. Multidimensional auctions are a generalization of single item auctions. These auctions can be classified:

- multi-item auction,
- multi-type auction,
- multi-round auction,
- multi-criteria auction.

Multi-item auctions can place bids on combinations of items, so called combinatorial auctions. There are several types of auctions (forward, reverse, and double). There is an effort to propose a general multi-type auction that covers all the types. In the iterative approach, there are multiple rounds of bidding and allocation and the problem is solved in an iterative and incremental way. Iterative combinatorial auctions are attractive to bidders because they learn about their rivals' valuations through the bidding process, which could help them to adjust their own bids.

Auctions with complex bid structures are also called multi-criteria auctions, since they address multiple attributes of the items (price, quality, quantity, etc.) in the negotiation space. Multi-criteria optimization can be helpful for detailed analysis of auctions.

There are possible combinations of the multidimensional characteristics. We propose a complex trading model based using of iterative process for multi-criteria combinatorial multi-type auctions.

3. Multi-item Auctions

There are many applications of combinatorial auctions. One of the most popular examples is using of combinatorial auctions for allocating radio spectrums for wireless communications services. Many types of combinatorial auctions can be formulated as mathematical programming problems. From different types of combinatorial auctions we present a forward auction of indivisible items with one seller and multiple buyers (Cramton, 2002; Sandholm, 2002). Let us suppose that one seller S offers a set R of r items, $j = 1, 2, \dots, r$, to n potential buyers B_1, B_2, ..., B_n.

Items are available in single units. A bid made by buyer B_i, $i = 1, 2, \dots, n$, is defined as

$$b_i = \{C, p_i(C)\}$$

where

$C \subseteq R$ – a combination of items,

$p_i(C)$ – the offered price by buyer B_i for the combination of items C.

The objective is to maximize the revenue of the seller given the bids made by buyers. Constraints establish that no single item is allocated to more than one buyer.

Binary variables are introduced for model formulation:

$x_i(C)$ is a binary variable specifying if the combination C is assigned to buyer B_i ($x_i(C) = 1$).

The forward auction can be formulated as follows

$$\sum_{i \in N} \sum_{C \subseteq R} p_i(C)\, x_i(C) \rightarrow \max$$

subject to
$$\sum_{i \in N} \sum_{C \subseteq R} x_i(C) \leqq 1, \ \forall j \in R \tag{8}$$
$$x_i(C) \in \{0, 1\}, \ \forall\ C \subseteq R, \ \forall i,\ i \in N = \{1, 2, \dots, n\}$$

The objective function expresses the revenue. The constraints ensure that overlapping sets of items are never assigned. The problem (8) is called the winner determination problem.

Complexity is a fundamental question in combinatorial auction design (Rothkopf et al., 1998). There are some types of complexity:

- computational complexity,
- valuation complexity,
- strategic complexity,
- communication complexity.

Computational Complexity covers the expected computation amount of the mechanism to compute an outcome given the bid information of the bidders. This is an extremely important question because winner determination problem is an NP-complete optimization problem.

Valuation complexity deals with the required computation amount to provide preference information within a mechanism. Estimating every possible bundle of items requires exponential space and hence exponential time. Bidders need to determine valuations for $2^m - 1$ possible bundle.

Strategic complexity concerns the best strategy for bidding. Which of the $2^m - 1$ bundles to bid on? Must bidders model behaviour of other bidders and solve problems to compute an optimal strategy?

Communication complexity concerns the required communication amount to exchange between bidders and auctioneer until an equilibrium price is reached. The problem of communication complexity can be addressed through the design of careful bidding languages that provide expressive but concise bids. Many researchers consider iterative auctions as an alternative.

4. Multi-type Auctions

An auction provides a mechanism for negotiation between buyers and sellers. In forward auctions a single seller sells resources to multiple buyers (model (8)). The forward auctions are typical for selling scarce or perishable items. In a reverse auctions, a single buyer attempts to source resources from multiple suppliers, as is common in industrial procurement (Bichler et al., 2006). Auctions with multiple buyers and sellers are called double auctions. Auctions with multiple buyers and sellers are becoming increasing popular. There are numerous applications of double auctions in electronic commerce (Wurman, Walsh and Wellman, 1998), including stock exchanges, business-to-business commerce, bandwidth allocation, etc. It is well known that double auctions in which both sides submit demand or supply bids are much more efficient than several one-sided auctions combined. Attention is devoted to double combinatorial auctions. Combinatorial double auctions can be transformed to combinatorial single-sided auctions and solved by methods for these auctions. Special case of double auction for one seller is the forward auction and special case of double auction for one buyer is the reverse auction.

We present a ***reverse auction*** of indivisible items with one buyer and several sellers. This type of auction is important for supplier selection problem. Let us suppose that m potential sellers S_1, S_2, ..., S_m offer a set R of r items, $j = 1, 2, \ldots, r$, to one buyer B.

A bid made by seller S_h, $h = 1, 2, \ldots, m$, is defined as

$$b_h = \{C, c_h(C)\}$$

where

$C \subseteq R$ – a combination of items,

$c_h(C)$ – the offered price by seller S_h for the combination of items C.

The objective is to minimize the cost of the buyer given the bids made by sellers. Constraints establish that the procurement provides at least set of all items.

Binary variables are introduced for model formulation:

$y_h(C)$ is a binary variable specifying if the combination C is bought from seller S_h ($y_h(C) = 1$).

The reverse auction can be formulated as follows

$$\sum_{h \in M} \sum_{C \subseteq R} c_h(C)\, y_h(C) \rightarrow \min$$

subject to
$$\sum_{h \in M} \sum_{C \subseteq R} y_h(C) \geq 1, \quad \forall j \in R \tag{9}$$
$$y_h(C) \in \{0, 1\}, \ \forall\, C \subseteq R, \ \forall h, h \in M = \{1, 2, \ldots, m\}$$

The objective function expresses the cost. The constraints ensure that the procurement provides at least set of all items.

Double auctions (auctions with multiple buyers and multiple sellers) are becoming increasing popular in electronic commerce. The numerous applications in electronic commerce, including stock exchanges, business-to-business commerce, bandwidth allocation, etc. have led to a great deal of interest in double auctions (see Bellosta et al., 2004).

For double auctions, the auctioneer is faced with the task of matching up a subset of the buyers with a subset of the sellers. The profit of the auctioneer is the difference between the prices paid by the buyers and the prices paid to the sellers. The objective is to maximize the profit of the auctioneer given the bids made by sellers and buyers. Constraints establish the same conditions as in single-sided auctions.

We present a double auction problem of indivisible items with multiple sellers and multiple buyers (see Xia, Stallaert and Whinston, 2005). Let us suppose that m potential sellers $S_1, S_2, ..., S_m$ offer a set R of r items, $j = 1, 2, \ldots, r$, to n potential buyers $B_1, B_2, ..., B_n$.

A bid made by seller S_h, $h = 1, 2, \ldots, m$, is defined as $b_h = \{C, c_h(C)\}$, a bid made by buyer B_i, $i = 1, 2, \ldots, n$, is defined as $b_i = \{C, p_i(C)\}$

where

$C \subseteq R$ – a combination of items,

$c_h(C)$ – the offered price by seller S_h for the combination of items C,

$p_i(C)$ – the offered price by buyer B_i for the combination of items C.

Binary variables are introduced for model formulation:

$x_i(C)$ – a binary variable specifying if the combination C is assigned to buyer B_i ($x_i(C) = 1$),

$y_h(C)$ – a binary variable specifying if the combination C is bought from seller S_h ($y_h(C) = 1$).

$$\sum_{i\in N} \sum_{C\subseteq R} p_i(C)\, x_i(- \sum_{h\in M} \sum_{C\subseteq R} c_h(C)\, y_h(C) \quad \rightarrow \quad \max$$

subject to
$$\sum_{i\in N} \sum_{C\subseteq R} x_i(C) \leqq \sum_{h\in M} \sum_{C\subseteq R} y_h(C),\ \forall j \in R \tag{10}$$
$$x_i(C) \in \{0, 1\},\ \forall\ C \subseteq R,\ \forall i,\ i \in N = \{1, 2, \ldots, n\}$$
$$y_h(C) \in \{0, 1\},\ \forall\ C \subseteq R,\ \forall h,\ h \in M = \{1, 2, \ldots, m\}$$

The objective function expresses the profit of the auctioneer. The constraints ensures for buyers to purchase a required item and that the item must be offered by sellers.

The formulated combinatorial double auction can be transformed to a combinatorial single-sided auction. Substituting $y_h(C)$, $h = 1, 2, \ldots, m$, with $1 - x_i(C)$, $i = n + 1, n + 2, \ldots, n + m$, and substituting $c_h(C)$, $h = 1, 2, \ldots, m$, with $p_i(C)$, $i = n + 1, n + 2, \ldots, n + m$, we get a model of a combinatorial single-sided auction.

$$\sum_{i\in NM} \sum_{C\subseteq R} p_i(C)\, x_i(C) - \sum_{i\in NM} \sum_{C\subseteq R} p_i(C) \quad \rightarrow \quad \max$$

subject to
$$\sum_{i\in NM} \sum_{C\subseteq R} x_i(C) \leqq m,\ \forall j \in R \tag{11}$$
$$x_i(C) \in \{0, 1\},\ \forall\ C \subseteq R,\ \forall i,\ i \in NM = \{1, 2, \ldots, n + \mathrm{m}\}$$

The model (11) can be solved by methods for single-sided combinatorial auctions. The specific forward or reverse auctions can be modelled as special cases of the model (11).

5. Multi-round Auctions

The key challenge in the iterative combinatorial auctions design is to provide information feedback to the bidders after each iteration (Pikovsky and Bichler, 2005; Parkes, 2006). Pricing was adopted as the most intuitive mechanism of providing feedback. Multi-round approaches are used in models for industrial procurement auctions (see Bichler et al., 2006). In contrast to the single-item single-unit auctions, pricing is not trivial for iterative combinatorial auctions. The main difference is the lack of the natural single-item prices. With bundle bids setting independent prices for individual items is not obvious and often even impossible. Different pricing schemes are introduced and discussed their impact on the auction outcome.

A set of prices $p_i(C)$, $i = 1, 2, \ldots, n$, $C \subseteq R$ is called:

- ***linear***, if $\forall\, i, C : p_i(C) = \sum_{j \in S} p_i(j)$
- ***anonymous***, if $\forall\, k, l, C : p_k(C) = p_l(C)$

Prices are linear if the price of a bundle is equal to the sum of the prices of its items, and anonymous if the prices of the same bundle are equal for every bidder. The simple pricing scheme with linear anonymous prices will be used. Linear anonymous prices are easily understandable and usually considered fair by the bidders. The communication costs are also minimized, because the amount of information to be transferred is linear in the number of items.

A set of prices $p_i(S)$ is called ***compatible*** with the allocation $x_i(C)$ and valuations $v_i(C)$, if

$$\forall\, i, C : x_i(C) = 0 \Leftrightarrow p_i(C) > v_i(C) \text{ and } x_i(C) = 1 \Leftrightarrow p_i(C) \le v_i(C)$$

The set of prices is compatible with the given allocation at the given valuations if and only if all winning bids are higher than or equal to the prices and all loosing bids are lower than the prices (assuming the bidders bid at their valuations).

Compatible prices explain the winners why they won and the losers, why they lost. In fact, informing the bidders about the allocation $x_i(C)$ is superfluous, if compatible prices are communicated. However, not every set of compatible prices provides the bidder with meaningful information for improving bids in the next auction iteration. Another important observation is the fact that linear compatible prices are harder and often even impossible to construct, when the bidder valuations are super- or sub-additive.

A set of prices $p_i(C)$ is in ***competitive equilibrium*** with the allocation $x_i(C)$ and valuations $v_i(C)$, if

1. The prices $p_i(C)$ are compatible with the allocation $x_i(C)$ and valuations $v_i(C)$.
2. Given the prices $p_i(C)$, there exists no allocation with larger total revenue than the revenue of the allocation $x_i(C)$.

The idea behind this concept is to define prices characterizing the optimal allocation. The prices may not be too low to violate the compatibility condition 1, but they may not be too high to violate the condition 2. In general, one can show that the existence of competitive equilibrium prices implies optimality of the allocation.

Primal-dual Algorithms

One way of reducing some of the computational burden in solving the winner determination problem is to set up a fictitious market that will determine an allocation and prices in a decentralized way. In the iterative approach, there are multiple rounds of bidding and allocation and the problem is solved in an iterative and incremental way. Iterative combinatorial auctions are attractive to bidders because they learn about their rivals' valuations through the bidding process, which could help them to adjust their own bids.

There is a connection between efficient auctions for many items, and duality theory. The Vickrey auction can be taken as an efficient pricing equilibrium, which corresponds to the optimal solution of a particular linear programming problem and its dual. The simplex algorithm can be taken as static approach to determining the Vickrey outcome. Alternatively, the primal-dual algorithm can be taken as a decentralized and dynamic method to determine the pricing equilibrium. A primal-dual algorithm usually maintains a feasible dual solution and tries to compute a primal solution that is both feasible and satisfies the complementary slackness conditions. If such a solution is found, the algorithm terminates. Otherwise the dual solution is updated towards optimality and the algorithm continues with the next iteration. The fundamental work (Bikhchandani and Ostroy, 2002) demonstrates a strong interrelationship between the iterative auctions and the primal-dual linear programming algorithms. A primal-dual linear programming algorithm can be interpreted as an auction where the dual variables represent item prices. The algorithm maintains a feasible allocation and a price set, and it terminates as the efficient allocation and competitive equilibrium prices are found.

For the winner determination problem we will formulate the LP relaxation and its dual. Consider the LP relaxation of the winner determination problem (8):

$$\sum_{i \in N} \sum_{C \subseteq R} v_i(C)\, x_i(C) \rightarrow \max$$

subject to

$$\sum_{C \subseteq R} x_i(C) \leq 1, \ \forall i, i \in N = \{1, 2, \ldots, n\}$$

$$\sum_{i \in N} \sum_{C \subseteq R} x_i(C) \leq 1, \ \forall j \in R \tag{12}$$

$$x_i(C) \geq 0, \ \forall C \subseteq R, \ \forall i, i \in N = \{1, 2, \ldots, n\}$$

The corresponding dual to problem (12)

$$\sum_{i \in N} p(i) + \sum_{j \in C} p(j) \to \quad \min$$

subject to

$$p(i) + \sum_{j \in C} p(j) \geq v_i(C) \ \forall\, i, C \tag{13}$$

$$p(i), p(j) \geq 0, \ \forall\, i, j$$

The dual variables $p(j)$ can be interpreted as anonymous linear prices of items, the term $\sum_{j \in S} p(j)$ is then the price of the bundle C and $p(i) = \max_S [v_i(C) - \sum_{j \in S} p(j)]$ is the maximal utility for the bidder i at the prices $p(j)$.

Following two important properties can be proved for the problems (12) a (13):

1. The complementary-slackness conditions are satisfied if and only if the current allocation (primal solution) and the prices (dual solution) are in competitive equilibrium.

2. The formulation (12) – (13) is weak. For the optimal allocation there no always exist anonymous linear competitive equilibrium prices.

Auction Formats

Several auction formats based on the primal-dual approach have been proposed in the literature. Though these auctions differ in several aspects, the general scheme can be outlined as follows:

1. Choose minimal initial prices.

2. Announce current prices and collect bids. Bids have to be higher or equal than the prices.

3. Compute the current dual solution by interpreting the prices as dual variables. Try to find a feasible allocation, an integer primal solution that satisfies the stopping rule. If such solution is found, stop and use it as the final allocation. Otherwise update prices and go back to 2.

Concrete auction formats based on this scheme can be implemented in different ways. The most important design choices are the following: bid structure, pricing scheme, price update rule, bid validity, feedback, way of computing a feasible primal solution in each iteration, and stopping rule.

6. Multi-criteria Auctions

Multi-criteria auctions allow negotiation on multiple criteria, involving not only the price, but also other criteria such as quality, guarantee, delivery terms and conditions. Multi-criteria approaches are used in models for electronic auctions (see Belosta et al., 2004). Multi-criteria auctions can be modelled as a multi-objective linear programming model

$$f_j(\mathbf{x}) = \sum_{i\in N} \sum_{C\subseteq R} v_{ij}(C)\, x_i(C) \rightarrow \quad \max, \quad j = 1, 2, \ldots, k$$

subject to
$$\sum_{C\subseteq R} x_i(C) \leq 1,\ \forall i,\ i = 1, 2, \ldots, n$$

$$\sum_{i\in N} \sum_{C\subseteq R} x_i(C) \leq 1,\ \forall j \in R \tag{14}$$

$$x_i(C) \geq 0,\ \forall\ C \subseteq R,\ \forall i,\ i \in N = \{1, 2, \ldots, n\}$$

where

$f_j(\mathbf{x})$, $j = 1, 2, \ldots, k$ – objective functions,

$\mathbf{x} \in X$ – a solution vector from the feasibility set X,

$v_{ij}(C)$ – the value of the criterion j for buyer B_i for the combination of items C.

The vector function of all objectives is denoted as $F(\mathbf{x})$.

We propose to solve the problem (14) by multi-round ALOP (Aspiration Level Oriented Procedure) (see Fiala, 1997). People appear to satisfy rather than attempting to optimize. That means substituting goals of reaching specified aspiration levels for goals of maximizing.

We denote $\mathbf{y}(t)$ the vector of aspiration levels of the objectives and $\Delta\mathbf{y}(t)$ changes of aspiration levels in the round t. The problem (14) can be substitute by a general aspiration level formulation

$$\begin{gathered} F(\mathbf{x}) \geq \mathbf{y}(t) \\ \mathbf{x} \in X \end{gathered} \tag{15}$$

According to heuristic information from results of the condition (15) the agent changes the aspiration levels of objectives for the round t + 1:

$$\mathbf{y}(t + 1)= \mathbf{y}(t) + \Delta\mathbf{y}(t) \tag{16}$$

There are three possibilities for aspiration levels $\mathbf{y}(t)$. The problem (15) can be feasible, infeasible or the problem has a unique non-dominated solution. We verify the three possibilities by solving the problem

$$\begin{gathered} v(\mathbf{d}^+) \rightarrow \min \\ F(\mathbf{x}) - \mathbf{d}^+ = \mathbf{y}(t) \\ \mathbf{x} \in X,\ \mathbf{d}^+ \geq 0 \end{gathered} \tag{17}$$

where

$\mathbf{d}^+$ – a vector of positive deviation variables,

$v(\mathbf{d}^+)$ – an objective function of $\mathbf{d}^+$.

The value of the objective function in the problem (17) can be interpreted as an increase of utility.

If it holds:

- $v > 0$, then the problem is feasible and $\mathbf{d}^+$ are proposed changes $\Delta\mathbf{y}(t)$ of aspiration levels which achieve a non-dominated solution in the next round;
- $v = 0$, then we obtained a non-dominated solution;
- the problem is infeasible, then we search the nearest solution to the aspiration levels by solving the goal programming problem

$$\begin{gathered} v(\mathbf{d}^+, \mathbf{d}^-) \rightarrow \min \\ F(\mathbf{x}) - \mathbf{d}^+ + \mathbf{d}^- = \mathbf{y}(t) \\ \mathbf{x} \in X, \mathbf{d}^+ \geq 0, \mathbf{d}^- \geq 0 \end{gathered} \tag{18}$$

The solution of the problem (18) is feasible with changes of aspiration levels $\Delta\mathbf{y}(t) = \mathbf{d}^+ - \mathbf{d}^-$. For changes of non-dominated solutions the duality theory can be applied (see Fiala, 1981). Dual variables to objective constraints in the problem (18) are denoted ui, $i \in K = \{1, 2, \ldots, k\}$.

If it holds

$$\sum_{i \in K} u_i \Delta y_i^{(t)} = 0 \tag{19}$$

then for some changes $\Delta\mathbf{y}(t)$ the value $v = 0$ is not changed and we obtain another non-dominated solution. The agent can state k-1 small changes of the aspiration levels $\Delta yi(t)$, $i \in K = \{1, 2, \ldots, k\}$, $i \neq r$, then the change of the aspiration level for criterion r is calculated from (19).

Results of the procedure ALOP are the path of tentative aspiration levels and the accepted non-dominated solutions.

Illustrative Example

The procedure is illustrated by the following numerical example of a double auction problem. Consider a simple supply chain. Let us suppose that 3 potential sellers S_1, S_2, S_3 offer a set R of 3 items $\{a, b, c\}$ to 3 potential buyers B_1, B_2, B_3. The auction is evaluated by two criteria.

First criterion is profit of the supply chain given as sum of differences between the offered prices by buyers and the offered prices by sellers.

Second criterion is delivery time for all 3 items $\{a, b, c\}$ and it should be minimised. The delivery time for all 3 items is a sum of delivery times for selected combinations of items. Offered prices for combinations of items are given in Table 1 and Table 2. Delivery times for combinations of items are given in Table 3.

Table 1

The Offered Prices by Sellers for Combinations of Items

	a	*b*	*c*	*ab*	*ac*	*bc*	*abc*
S_1	5*	10	15	12■	19	24	29
S_2	6	9	12■	14	17	19*	26
S_3	7	8	16	14	22	23	30

Source: Author.

Table 2

The Offered Prices by Buyers for Combinations of Items

	a	*b*	*c*	*ab*	*ac*	*bc*	*abc*
B_1	7	8	16	15	23	24	31
B_2	6	10	18■	16	24	28*	34
B_3	8*	9	15	17■	23	24	32

Source: Author.

Table 3

Delivery Times for Combinations of Items

	a	*b*	*c*	*ab*	*ac*	*bc*	*abc*
S_1	8*	7	6	8■	11	13	20
S_2	8	7	8■	8	9	10*	18
S_3	5	10	11	10	13	16	19

Source: Author.

The problem was solved by the multi-round ALOP procedure. In the first round, the supply chain manager sets aspiration levels $\mathbf{y}(1) = (12, 15)$. There is no solution for these aspiration levels. In the second round, the aspiration levels were changed $\mathbf{y}(2) = (12, 18)$. The procedure found a solution (marked with asterisks * in Tables):

The seller S_2 sells the items $\{b, c\}$ to the buyer B_2 and the seller S_1 sells the item $\{a\}$ to the buyer B_3. The objective function values are

$$f_1(\mathbf{x}) = (28 - 19) + (8 - 5) = 12$$
$$f_2(\mathbf{x}) = 10 + 8 = 18$$

The supply chain manager is not satisfied with the delivery time. In the third round the aspiration levels were changed $\mathbf{y}(3) = (11, 16)$. The procedure found a solution (marked with squares ■ in Tables):

The seller S_1 sells the items $\{a, b\}$ to the buyer B_3 and the seller S_2 sells the item $\{c\}$ to the buyer B_2. The objective function values are

$$f_1(\mathbf{x}) = (17 - 12) + (18 - 12) = 11$$
$$f_2(\mathbf{x}) = 8 + 8 = 16$$

The supply chain manager is satisfied with the objective function values and the procedure stops.

Conclusions

Auctions are the important subject of an intensive economic research. The paper proposes a complex trading multi-type model based on multi-criteria iterative combinatorial auctions. A possible flexible approach for modelling and solving such auctions is presented. It is a combination of auction models – multi-item, multi-type, multi-round, multi-criteria. Individual auction models help the bidders express their preferences. Allowing bidders more fully to express preferences often leads to improved economic efficiency and greater auction revenue. The combination of such models can give more complex views on auctions.

The proposed model is general, including all components of multidimensional auctions together. The specific models with only some characteristics of multidimensional auctions can be simply derived from this general model. These specific models have many real applications.

The proposed solution procedure is possible to use for solving the general model. The procedure is oriented on changes of aspiration levels of objective function values, what is a natural approach to solving such problems. This approach is friendly and easy to understand for users. A simple illustrative example of the general model solved by the procedure was presented.

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